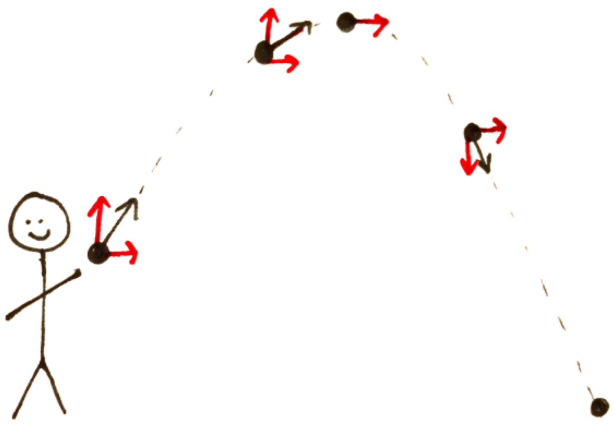
PROJECTILE MOTION

CONTENT

Projectile motion refers to the motion of a particle or object that is influenced by the acceleration due to the Earth's gravity (if we assume there is no air resistance). For example, throwing a ball in the air. Just like in kinematics, we can resolve the velocity of the projectile into its x and y components. (You can revise this in the Kinematics worksheet: Vector Components).

In the example of the ball, once the ball leaves your hands the only acceleration is downwards due to gravity. This means there is no horizontal acceleration. Since there is no horizontal acceleration, there is a constant horizontal velocity. The projectile is a parabola, as shown below. The black vector is the total velocity and the red vectors are the x and y components. Notice how the red horizontal vector doesn't change at different places despite the overall velocity changing.



We can calculate several things from the path of the projectile using the equations of motion. (You can revise these in the Kinematics worksheet: Equations of Motion).

$$s = ut + \frac{1}{2}at^2$$
$$v^2 = u^2 + 2as$$
$$v = u + at$$

For each of these equations, we can resolve the displacement, s, the initial velocity, u, the final velocity, v, and the acceleration, a, into their x and y components. This is how to derive the equations for projectile motion. The acceleration in the x-direction is always zero and the acceleration in the y-direction is always due to gravity. So, the acceleration doesn't change with time. This is called **uniform acceleration**.

For example, we can resolve the displacement into the x and y displacement in the first equations in the following way:

$$u_x = u\cos\theta \qquad u_y = u\sin\theta$$
$$v_x = v\cos\theta \qquad v_y = v\sin\theta$$
$$a_x = 0 \qquad a_y = -g$$

$$s = ut + \frac{1}{2}at^2 \qquad s = ut + \frac{1}{2}at^2$$
$$s_x = u\cos\theta\, t \qquad s_y = u\sin\theta\, t - \frac{1}{2}gt^2$$

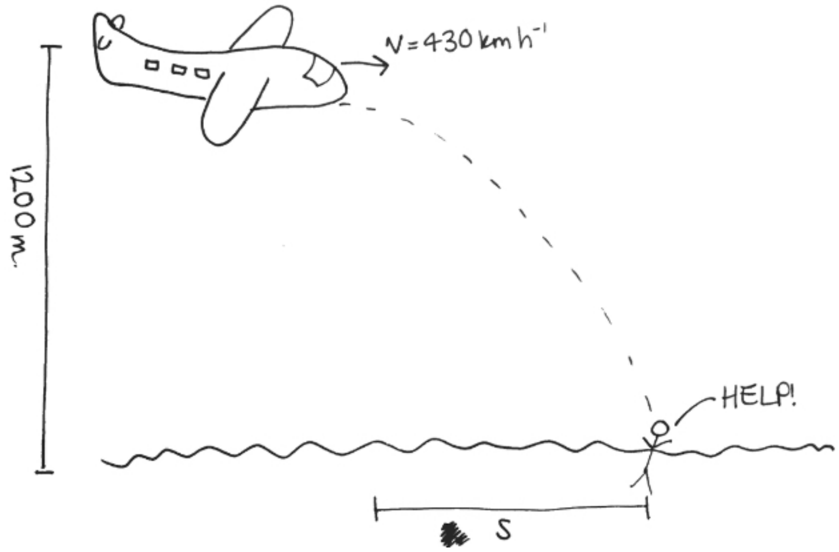
Just like in kinematics, the best way to approach problems is:

1. Draw a diagram of the problem deciding which direction is positive and which is negative
2. Write down all the variables we know and what we're looking for
3. Determine what equation to use to solve the problem

EXAMPLE

A rescue plane is flying at constant elevation of 1200 m with a speed of 430 kmh^{-1} toward a point directly above a person struggling in the water. At what distance should the pilot release a rescue capsule if it is to strike close to the person in the water?

⇒ So firstly, we will draw a diagram of the problem setting downwards as negative and upwards as positive with the origin set at the plane:



⇒ Now, to write down all the variables we have and determine which formula we should use noting, the capsule is released straight ahead so there is no angle, the only acceleration is due to gravity and we are ignoring air resistance:

Variable	Value
s_y	$-1200m$
u_x	$119.4ms^{-1}$
u_y	$0ms^{-1}$
θ	$0°$
a_y	$-9.8ms^{-1}$
s_x	?

Since we need to find s_x first we will need to solve $s = ut + \frac{1}{2}at^2$ in the y direction for t. Then we will sub that value into the same equation so solve for s_x.

⇒ So, finally calculating:

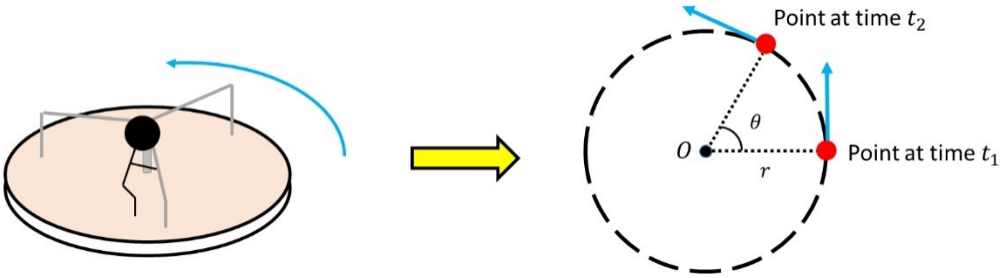
$$s = ut + \frac{1}{2}at^2$$
$$s_y = u_y t + \frac{1}{2}at^2$$
$$-1200 = 0.t + \frac{1}{2}(-9.8)t^2$$
$$-1200 = -4.9t^2$$
$$t^2 = \frac{1200}{4.9}$$
$$\Rightarrow t = 15.649\ldots$$

$$s = ut + \frac{1}{2}at^2$$
$$s_x = u_x t + \frac{1}{2}a_x t^2$$
$$s_x = 119.4 \times 15.65 + \frac{1}{2}(0)(15.65)^2$$
$$= 1869.21\ldots$$
$$\Rightarrow s_x = 1869m$$

CIRCULAR MOTION

CONTENT – UNIFORM CIRCULAR MOTION

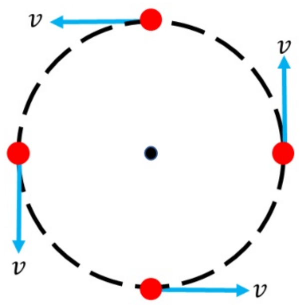
In module 1 we looked at the motion of an object with mass in a straight line. Here in module 5, we will look at the motion of an object moving in a circular path. An example where you might have experienced moving in a circular motion is the Merry-Go-Round, and we *feel* a 'force' pushing us away from the centre. Why do we *feel* this 'force'? Let's use physics to understand this problem and assume that the motion can be represented on a 2D plane with the centre located at the origin O. We will represent the person riding the Merry-Go-Round as an object with *mass* (m) positioned distance (r) away from the origin (i.e. *radius*). If the Merry-Go-Round moves anti-clockwise, then we can represent the motion as



Average speed is equal to distance divided by time, and for circular motion, the total distance travelled is equal to the circumference of the circle ($C = 2\pi r$). The total time it takes for the object to return to its original position is called the period T. Thus the average speed of the object moving in a circle with radius r is

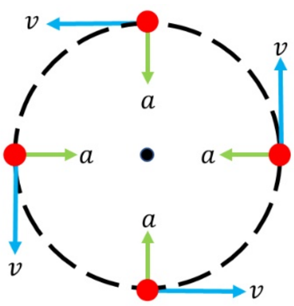
$$|v| = \frac{2\pi r}{T} \quad (1)$$

The velocity of the object at any point on the circumference is equal to the instantaneous speed at that point. The direction of the velocity follows the same path as the motion of the object. Since the motion is circular, the direction of the velocity will change continuously as shown on the right.



It is important to note that although the velocity of the object continuously changes (i.e. direction) the average speed (i.e. the magnitude of the velocity) is the same in uniform circular motion. We can quantify the angular velocity by dividing the change in angle, $\Delta\theta = \theta_2 - \theta_1$, between two points on the circular path over time. In terms of linear velocity the angular velocity is equal to the tangential velocity diveded by the radius.

$$\omega = \frac{\Delta\theta}{t} = \frac{v}{r} \quad (2)$$



For an object in uniform motion, the acceleration of an object is equal to zero as any change in acceleration will change the velocity. This is also true for uniform circular motion; if the acceleration in the direction of the velocity is not zero, then the motion will not be uniform. However, an object moving in a circle does have an acceleration called the '*centripetal acceleration*'. The centripetal acceleration points in the direction of the centre of the circle (perpendicular to the velocity vector). Since the direction is perpendicular to the velocity vector, the average acceleration does not change. This is demonstrated on the image on the left. The equation for centripetal acceleration is given by

$$\vec{a} = \frac{|\vec{v}|^2}{r} \quad (3)$$

Recall that the force of an object given by Newton's 2nd law is $F = ma$. Using this equation, we can calculate the '*centripetal force*' of an object moving in a circular motion in the same direction of the centripetal acceleration.

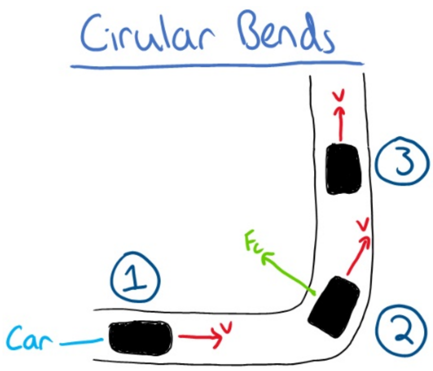
$$\vec{F} = m\frac{|\vec{v}|^2}{r} \quad (4)$$

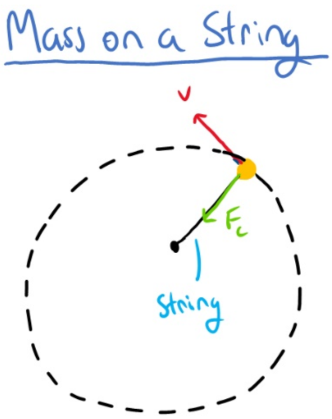
So why do we feel a 'force' pushing us away from the centre? We feel we are being pushed outwardly because the velocity vector is tangential to the circular path. Our body wants to move in a straight path, but since we are holding on the bar of the Merry-Go-Round, there is a centripetal force pointing towards the centre. This prevents us flying out of the Merry-Go-Round, unless of course until we let go of the bar.

CONTENT – REAL WORLD EXAMPLES

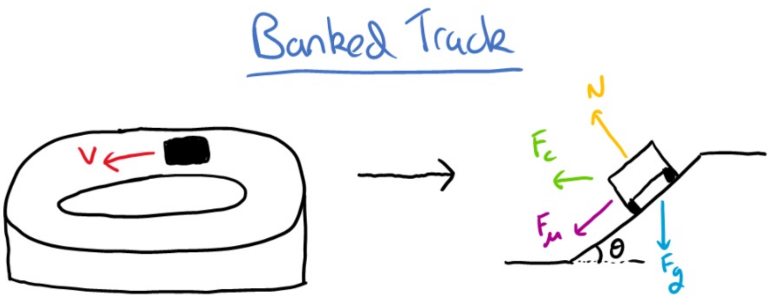
For a car moving around a circular bend the car will experience certain forces. Following the labels on the diagram on the right:

1) The car moves in a straight line to the right with velocity v.
2) As the car makes a turn on the circular bend, the car experiences a centripetal force due to the friction between the tyres and the surface of the road. The passenger inside the car is pushed outwardly in the opposite direction to the centripetal force. The direction of the velocity changes.
3) After the turn the car moves in a straight path again and the centripetal force vanishes.





The example on the left is of a mass attached to a string is similar to the Merry-Go-Round example. As we spin the string the mass follows a circular path. The velocity is in the direction of the motion and tangential to the path. The mass at the end of the string experiences a centripetal force pointing to the other end of the string. The centripetal force is a result of the mass attached to the string. If the string breaks the object will fly off the circular path.



The last example on the right is of an object moving on a banked track. The diagram next to the track shows the forces available on the object. The centripetal force is a result of the sum of the frictional F_μ and normal F_N force.

$$F_c = F_\mu + F_N$$

CIRCULAR MOTION – ENERGY AND WORK

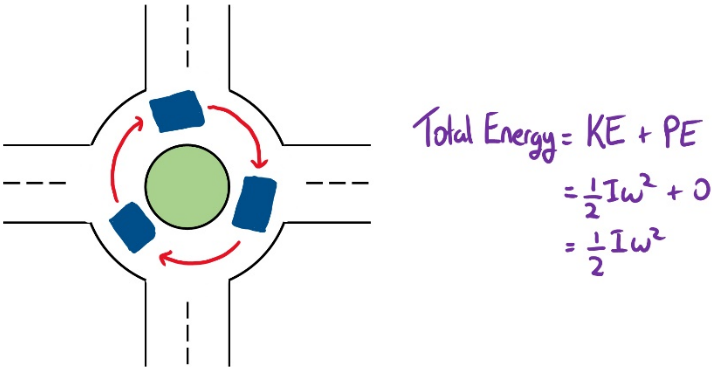
CONTENT – TOTAL ENERGY

We have learned the concept of kinetic energy in *Module 2: Dynamics* and it is given by $\frac{1}{2}mv^2$. The velocity in the equation is for linear velocity. Angular velocity is linear velocity divdied by the radius - $\omega = v/r$. Rearranging this equation to the make linear velocity on the left-hand side and substituting into the kinetic energy expression we get

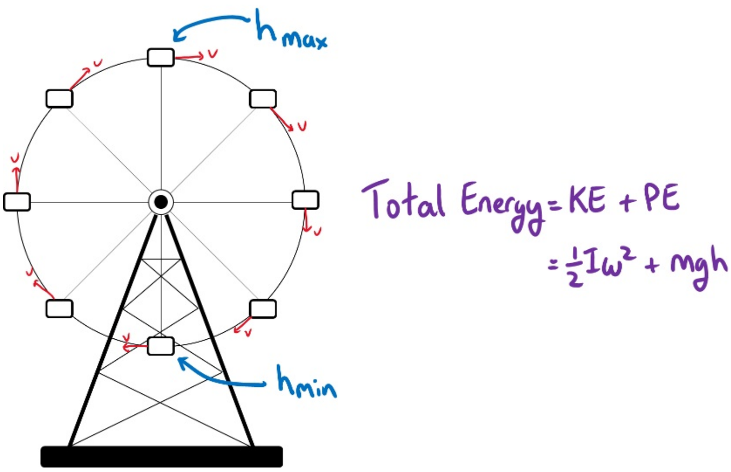
$$\begin{aligned} KE &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m(\omega r)^2 \\ &= \frac{1}{2}m\omega^2 r^2 \\ &= \frac{1}{2}I\omega^2 \end{aligned} \tag{1}$$

In the last step we have replaced mr^2 with the variable I. This quantity is called the *moment of inertia* or *rotational inertia* and is a measure of how much an object resists rotational motion.

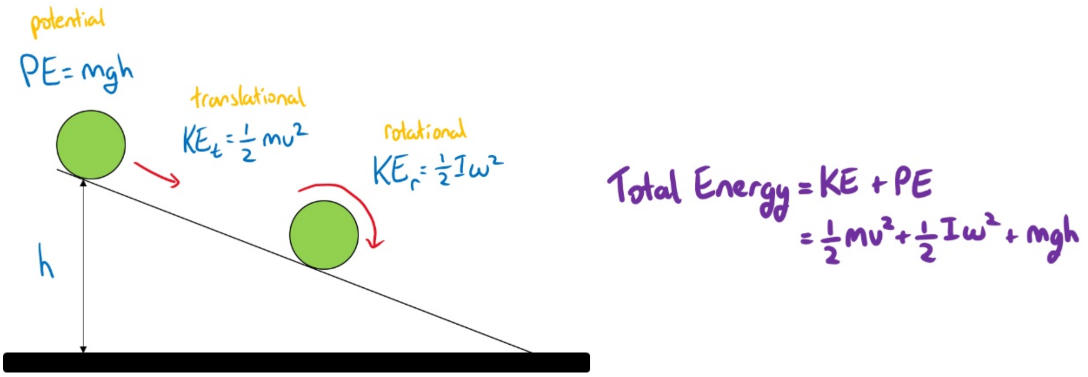
The potential energy of a body is given by $PE = mgh$, which depends on the height. For a circular motion on a flat surface, like a car turning in a roundabout, the height can be considered zero resulting in a zero potential energy. Thus, the total energy for an object in uniform circular motion on a flat horizontal surface is just the rotational kinetic energy as shown in Figure (A) below.



If the circular motion varies with height then the potential energy is not necesarrily zero. Figure (B) shows a Ferris wheel with 8 carrieges. The total energy is equal to the rotational kinetic energy plus the potential energy. The potential energy at the bottom, carriege 5, is at the minimum while at the top, carriege 1, the potential energy is at the maximum. Thus, the total energy of a carriege changes along the circular path.

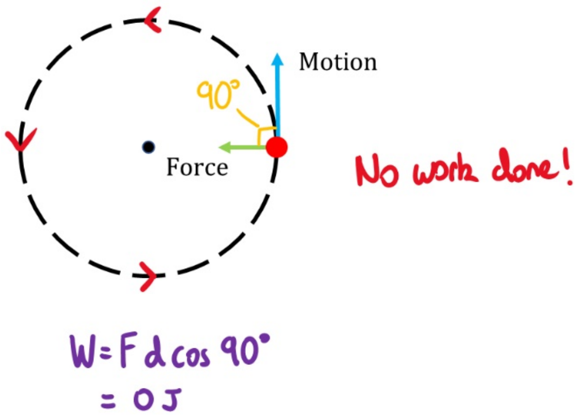


As a final example, consider a ball rolling down a frictionless inclined surface, Figure (C). The ball initially has a potential energy of mgh and as the ball rolls this potential energy is converted into kinetic energy. The ball will move with both translational kinetic energy, $\frac{1}{2}mv^2$, and rotational kinetic energy, $\frac{1}{2}I\omega^2$. Therefore, the total energy of the ball is equal to the sum of both kinetic term and the potential energy.



CONTENT – WORK

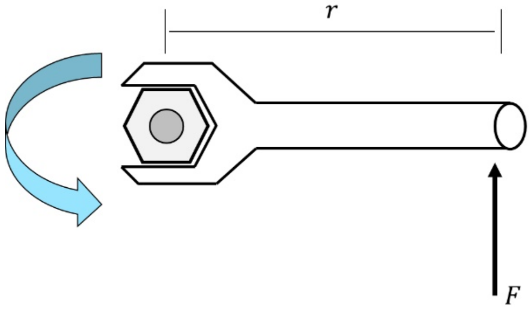
Work is defined as the amount of energy required to move an object from point A to B by a given applied force (i.e. $W = F.d\cos\theta$). If the displacement is zero then the amount of work done is zero no matter how much force is applied to the object. In uniform circular motion the centripetal force is pointing towards the center and the displacement is perpendicular to the force. This means that the angle θ is $90°$, thus the amount of work is zero (i.e $\cos 90° = 0 \rightarrow F = 0$). Alternatively, we can arrive at the same conclusion if we define the work done as the change in kinetic energy (i.e. $W = \Delta KE$). If the object moves with constant velocity around the circular path then the change in kinetic energy is zero, leading to no work being done.



CIRCULAR MOTION – ROTATION AND TORQUE

CONTENT – TORQUE

We will now look at the rotation of a mechanical object where previously we have only looked at the circular motion of a simple object. Let's say we want to loosen a bolt using a wrench. We apply a force on the wrench anti-clockwise direction. If enough force is applied the bolt will be loosen. The tendency for the applied force to cause the rotational motion of the bolt is called *torque*. Torque depends not only of the applied force but also the distance the force is applied to from a pivot. The equation for torque in vector and scalar form is

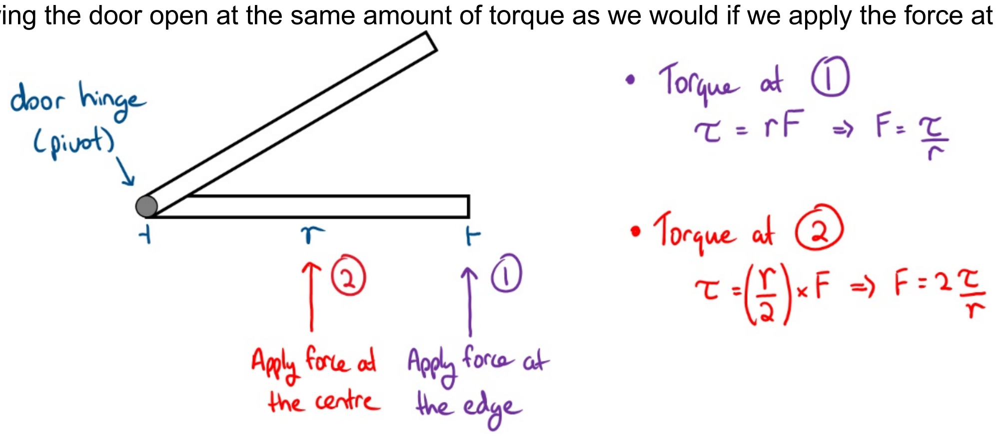


$$\tau = \vec{r}\vec{F}_{\perp} = |\vec{r}||\vec{F}|\sin\theta \qquad (1)$$

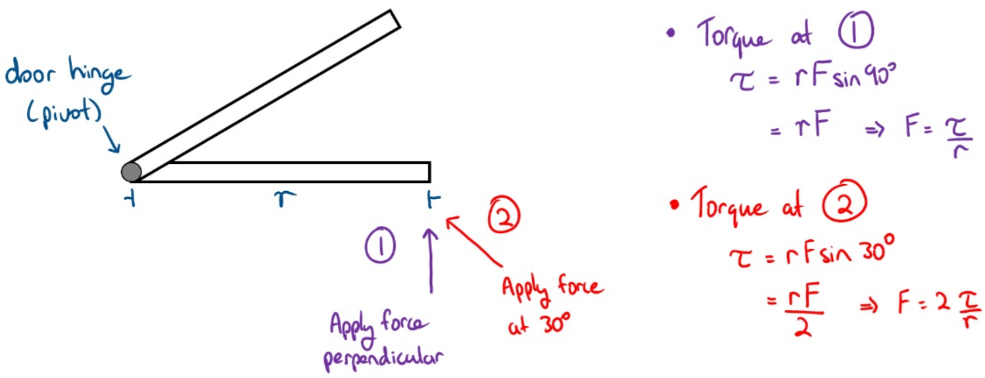
where *r* is the distance from the pivot point and *F* is the applied force. The quantity τ has units of Newton-metre *N.m* or Joule per radian *J/rad*.

REAL WORLD EXAMPLE – OPENING A DOOR

Using the definition in equation (1) we can learn a few things about the rotation of mechical objects. Let's say we want to open the door by pushing at a location half way between the hinges and the handle. We apply a perpendicular force on this point (i.e. $\theta = 90°$), which will make the sine term equal to one. The amount of force required to swing the door open at the same amount of torque as we would if we apply the force at the handle is



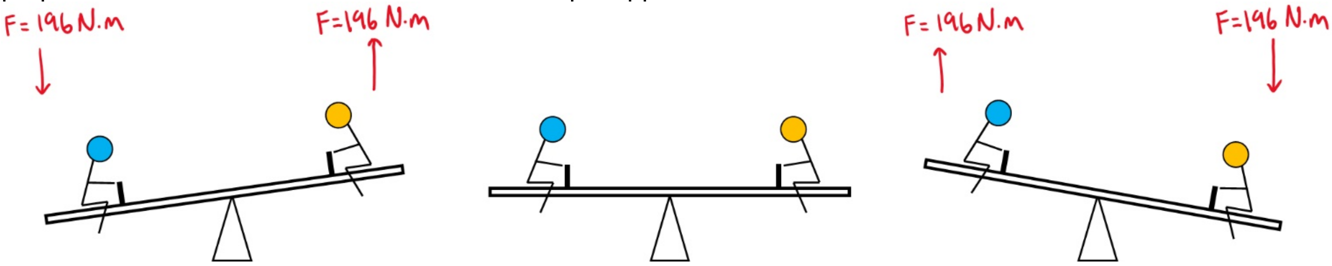
This means that we need to apply two times the amount force on the same door! Another situation we can learn about torque with opening a door is when you apply the force that is not perpendicular. Let's say we apply the force that is $30°$ to the surface of the door at the handle. The amount of force required to swing the door will be two times as well.



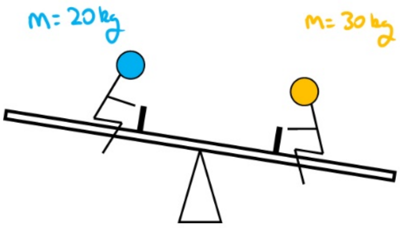
Thus, the reason why doorknobs/handles are located near the edge opposite from the hinges and apply the force perpendicularly is so we only need minimal effort to swing the door.

Real World Example – Seesaw

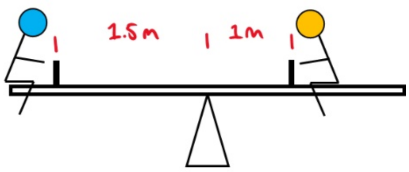
Another real world example where we see torque in action is the seesaw in the park. The seesaw is pivoted at the centre (fulcrum) and as one end is lifted the other end goes down. Suppose two children are sitting 1 *m* away from the fulcrum with masses of 20 *kg*. The amount of force applied on each end is the same (*F*=mg=196 *N*) and is perpendicular to the seesaw. The amount of torque applied on each side is 196 *N.m*.



Suppose one of the children is 30 *kg*, this will create an imbalance in the seesaw. To restore the balance in the seesaw the with the lower mass needs to move on the seesaw. To determine the location we make the torque of each side to be equal. Then we have one unknown in the equation, which is the distance the child must sit.



$$\tau_1 = \tau_2$$
$$r_1 F_1 = r_2 F_2$$
$$r_1 m_1 g = r_2 m_2 g$$
$$r_1 = r_2 \frac{m_2}{m_1} = 1\,m \times \frac{30\,kg}{20\,kg} = 1.5\,m$$



Thus, the child must sit 1.5 *m* away from the fulcrum for the seesaw to be balanced.

Question 1

What is the torque on a bolt when a 150 *N* force is applied to a wrench of length 20 *cm*. The force is applied at an angle of 40º to the wrench.

Ans: 19.28 *N.m*

Question 2

Determine the angle at which the force is applied to lever if the force applied is 400 *N* resulting in a torque of 50 *N.m*. The force applied is located at 1.2 *m* from the pivot point.

Ans: 5.98º

GRAVITATIONAL MOTION 1

CONTENT

A gravitational field is the region surrounding a mass that attracts other bodies to it due to the force of gravity. The more massive the object, the greater its gravitational attraction. For example, the Earth has a far greater gravitational pull than a tennis ball. We can calculate the strength of a gravitational field using the equation:

$$g = \frac{GM}{r^2}$$

where g is the gravitational field strength, M is the mass that is producing the gravitational field in kg, r is the distance from the mass and G is the universal gravitational constant 6.67×10^{-11} Nm^{2}kg^{-2}. This gravitational field strength is also known as the acceleration due to gravity, it has units ms^{-2}. The gravitational field is isotropic, i.e. it is the same in all directions. It depends only on how massive the object is and how far away from the object you are. So being close to a very massive object will mean there is a large gravitational attraction.

When we put two objects near each other, they both have their own gravitational field. So, they are both experiencing a force of attraction to the other masses. Just like in Newton's Second Law which states $F = ma$, we can calculate the strength of the force of attraction between two masses, M and m, due to gravity using the same formula where my acceleration is the acceleration due to gravity calculated above:

$$F = \frac{GMm}{r^2}$$

where F is the force due to gravity, M and m are the masses of the two objects, r is the distance between them and G is the universal gravitational constant again. This means that any two masses experience a force of attraction due to gravity.

But, in both cases, the magnitude of that field and the force is tiny until we get to incredibly large masses like the Moon and the Earth. The strength increases for both if we increase the masses involved and decrease the distance.

EXAMPLE

Daliah is a space explorer who is tasked with comparing the gravitational field strength in different areas around the Solar System. She compares the strength of the gravitational field due to the Earth at the orbit of the Moon and the strength of the gravitational field due to Saturn at the orbit of one of its moons Rhea. Given the mass of the Earth is 6×10^{24}kg, the mass of Saturn is 568×10^{24}kg, the radius of the moons orbit is $385{,}000$km and the radius of Rheas orbit is $527{,}000$km, what are the factors that will affect the strength of the gravitational fields? Which location/position would Daliah measure a stronger gravitational field and why?

⇒ Firstly, we will explain what factors influence the strength of a gravitational field and then we will calculate the field in both positions in order to compare the two.

⇒ So, according to the equation for gravitational field strength, $g = \frac{GM}{r^2}$, the two variables that will influence the strength of the field are the mass of the object and the distance from that object. Since Saturn has a much larger mass than the Earth this will increase the field strength. However, the radius of Rheas orbit is also much larger than the orbit of the Moon. Since the gravitational field strength is inversely proportional to the distance squared from the mass this has a larger influence on the strength of the field. In this particular case, however, while the radius of Rheas orbit is larger than the Moon's the difference is nowhere near as large as the difference between the masses of Earth and Saturn. Thus, it is likely that the mass of Saturn compared to the mass of the Earth will play the dominant role in the gravitational field strength.

⇒ Now, to test our hypothesis we shall calculate the two gravitational fields at the orbits of their moons and compare:

$$g_E = \frac{GM_E}{r_m^2} = \frac{6.67\times10^{-11}\times6\times10^{24}}{(385{,}000\times10^3)^2} = 0.0027\ Nkg^{-1} = 0.0027ms^{-2}$$

$$g_S = \frac{GM_S}{r_R^2} = \frac{6.67\times10^{-11}\times568\times10^{24}}{(527{,}000\times10^3)^2} = 0.14Nkg^{-1} = 0.14ms^{-2}$$

⇒ So, the gravitational field strength due to Saturn at the orbit of its moon Rhea is larger than the gravitational field strength due to the Earth at the Moons orbit.

EXAMPLE

After comparing the strength of the gravitation field at Rhea's orbit and the Moon's orbit, Daliah decided it was also a good idea to determine the force between each planet and its satellite. Given the mass of the Moon is 73.5×10^20kg and the mass of Rhea is 2.31×10^{20}kg how does the force due to gravity between Saturn and Rhea compare with the force due to gravity between Earth and the Moon?

⇒ To solve this problem, first we will write down all the variables we have and then sub them into the equations. Then we will compare the two results:

Variable	Value
M_E (Mass of the Earth)	$6 \times 10^{24} kg$
M_S (Mass of Saturn)	$568 \times 10^{24} kg$
r_M (Radius of the Moons orbit)	$3.85 \times 10^{8} m$
r_R (Radius of Rheas orbit)	$5.27 \times 10^{8} m$
m_M (Mass of the Moon)	$7.35 \times 10^{22} kg$
m_R (Mass of Rhea)	$2.31 \times 10^{21} kg$

⇒ Now we can calculate the force between the two planets and their moons using the formula $F = \frac{GMm}{r^2}$:

$$F_{E\&M} = \frac{GM_E m_m}{r_m^2} = \frac{6.67\times10^{-11}\times6\times10^{24}\times7.35\times10^{22}}{(3.85\times10^8)^2} = 1.98\times10^{20}N$$

$$F_{S\&R} = \frac{GM_S m_R}{r_R^2} = \frac{6.67\times10^{11}\times568\times10^{24}\times2.31\times10^{21}}{(5.27\times10^8)^2} = 3.15\times10^{20}N$$

GRAVITATIONAL MOTION 2

CONTENT

The massive objects will experience a force of attraction due to their gravitational fields. This force is what keeps planets in orbits around the Sun and moons in orbit around their host planets. Just as in circular motion, we can calculate the velocity of planets orbiting around a host star using a very similar formula:

$$v = \frac{2\pi r}{T}$$

where r is the radius of the orbit, and T is the time taken for one full cycle of the orbit. This formula is for the average velocity of an object moving in a circle. While the orbits of planets are not perfectly circular, we can still approximate their orbits as circular and calculate the average velocity of their entire orbit using this formula.

Just like an object moving on the surface of the Earth, objects in orbit such as planets or satellites, have kinetic energy and gravitational potential energy. Potential energy on Earth is dependent on the mass of the object and the height/distance from the Earth. Gravitational potential energy is very similar:

$$U = -\frac{GMm}{r}$$

where M and m are the two masses in kg, r is the distance between them, or the radius of the orbit, and G is the universal gravitational constant 6.67×10^{-11}Nm^{2}kg^{-2}. While this is just the gravitational potential energy, the total energy of the satellite in orbit is a combination of the gravitational potential energy and the orbital kinetic energy. It can be found using the equation:

$$E = -\frac{GMm}{2r}$$

Both these equations for the energy of the satellite or planet are very similar and combine the mass and distance of the two objects involved. This means satellites that are more massive and orbit with a smaller radius have larger gravitational potential energy and total energy.

EXAMPLE

Given the Earth has a mass of 5.972×10^{24}kg and orbits the Sun each year at an average radius of 145×10^9m, what is the average velocity of the Earth's orbit and the total energy of the orbit, assuming the Sun's mass is 1.989×10^{30}kg?

⇒ We start by writing down all the variables we have and calculating the number of seconds in a year:

Variable	Value
m_e	5.972×10^{24}kg
M_s	1.989×10^{30}kg
r	145×10^9m
T	3.15×10^7s

⇒ Now we can sub these values into the equation for velocity and energy:

$$v = \frac{2\pi r}{T} = \frac{2 \times \pi \times 145\times10^9}{3.15\times10^7} = 28.9\times10^3 \text{ m/s}$$

$$E = -\frac{GMm}{2r} = -\frac{6.67\times10^{-11} \times 1.989\times10^{30} \times 5.972\times10^{24}}{2 \times 145\times10^9} = -2.73\times10^{33}\text{ J}$$